

Constraining the double gluon distribution by the single gluon distribution

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We show how to consistently construct initial conditions for the QCD evolution equations for double parton distribution functions in the pure gluon case. We use to momentum sum rule for this purpose and a specific form of the known single gluon distribution function in the MSTW parameterization. The resulting double gluon distribution satisfies exactly the momentum sum rule and is parameter free. We also study numerically its evolution with a hard scale and show the approximate factorization into product of two single gluon distributions at small values of x , whereas at large values of x the factorization is always violated in agreement with the sum rule.

Keywords: quantum chromodynamics, parton distributions, evolution equations, sum rules

I. INTRODUCTION

Multiparton interactions play an important role in the hadronic collisions at high energies. They occur when at one encounter of the initial hadrons, more than one partonic interaction occurs. They were first observed and measured at the Tevatron [1–4] and subsequently a systematic experimental study was performed at the higher energy Large Hadron Collider [5–7]. The theoretical description of such interactions within perturbative QCD is possible in the presence of the sufficiently hard scales. The computation of double parton scattering (DPS) cross sections within the collinear framework makes use of the double parton distribution functions (DPDFs) [8–35]. In the collinear leading logarithmic approximation DPDFs obey QCD evolution equations [8, 9, 12, 13, 18, 36], similar to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations for single parton distribution functions (PDFs). The evolution equations for DPDFs conserve new sum rules which relate the double and single parton distributions once they are imposed on initial conditions for the evolution equations at some initial scale. All the attempts up till now to construct conditions which satisfy these sum rules were rather unsuccessful, see e.g. Refs. [13, 18, 37] with an exception of the analysis [38] for valence quarks only.

In this letter, we show how to consistently perform such a construction in a pure gluon case, using the known single PDFs in the MSTW parameterization [39] and the momentum sum rule. We find the parameter free double gluon distribution which we evolve with our numerical program. In particular, we study the build up of its approximately factorizable form for small values of parton momentum fractions, $x_{1,2} < 0.1$. The full case with quarks and gluons is postponed to a separate publication.

II. EVOLUTION EQUATIONS AND SUM RULES

We consider the DPDFs with equal hard scales, $Q_1 = Q_2 \equiv Q$, and the relative momentum $\mathbf{q} = 0$:

$$D_{f_1 f_2}(x_1, x_2, Q) \equiv D_{f_1 f_2}(x_1, x_2, Q, Q, \mathbf{q} = 0) , \quad (1)$$

where $x_{1,2} \in [0, 1]$ are parton momentum fractions, which obey the condition $x_1 + x_2 \leq 1$, and $f_{1,2}$ are parton flavors (including gluon) [17, 22]. In this case, the evolution equations in the leading logarithmic approximation read

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} D_{f_1 f_2}(x_1, x_2, Q) = \frac{\alpha_s(Q)}{2\pi} \sum_{f'} \left\{ \int_{x_1}^{1-x_2} \frac{du}{u} \mathcal{P}_{f_1 f'}\left(\frac{x_1}{u}\right) D_{f' f_2}(u, x_2, Q) \right. \\ + \int_{x_2}^{1-x_1} \frac{du}{u} \mathcal{P}_{f_2 f'}\left(\frac{x_2}{u}\right) D_{f_1 f'}(x_1, u, Q) \\ \left. + \frac{1}{x_1 + x_2} P_{f' \rightarrow f_1 f_2}^R\left(\frac{x_1}{x_1 + x_2}\right) D_{f'}(x_1 + x_2, Q) \right\}. \end{aligned} \quad (2)$$

where the functions \mathcal{P} on the r.h.s. are the leading order Altarelli-Parisi splitting functions (with virtual corrections for \mathcal{P}_{ff} included). The third term on the r.h.s corresponds to the splitting of one parton into two daughter partons, described by the Altarelli-Parisi splitting function for real emission, $P_{f' \rightarrow f_1 f_2}^R$. It contains the single PDFs, $D_{f'}$, thus eq. (2) has to be solved together with the ordinary DGLAP equations, see e.g. Ref. [18] for more details.

The significance of the splitting terms in the evolution equations (2) for the computation of the double parton scattering cross sections was a subject of intensive debate in the literature over the last few years [17, 18, 21, 22, 26, 31, 32, 40]. The conclusion which emerges from this discussion is that the processes which are summed up by the splitting terms and coming from both hadrons in hadron-hadron collisions should rather be classified as the single parton scattering process [26]. On the other hand, the so called single splitting contributions, with parton splitting from one hadronic side only, are important for the double parton scattering cross sections [15, 31, 32, 41]. From the perspective of the present paper, in which we only concentrate on the evolution of the DPDFs, the splitting terms in the evolution equations are crucial for the conservation of sum rules which are discussed below.

The sum rules which are conserved by the evolution equations (2) are the momentum and valence quark number sum rules [14]. Imposing them for initial conditions specified at some initial scale Q_0 , they are guaranteed to be satisfied at any other scale Q . The momentum sum rule for the DPDFs reads

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 D_{f_1 f_2}(x_1, x_2) = (1 - x_2) D_{f_2}(x_2), \quad (3)$$

while the valence quark number sum rule is given by

$$\int_0^{1-x_2} dx_1 \{D_{q f_2}(x_1, x_2) - D_{\bar{q} f_2}(x_1, x_2)\} = (N_q - \delta_{f_2 q} + \delta_{f_2 \bar{q}}) D_{f_2}(x_2), \quad (4)$$

where $q = u, d, s$ and $N_u = 2, N_d = 1, N_s = 0$ are the valence quark number for each of the quark flavors. The same relations hold true with respect to the second parton

$$\sum_{f_2} \int_0^{1-x_1} dx_2 x_2 D_{f_1 f_2}(x_1, x_2) = (1 - x_1) D_{f_1}(x_1), \quad (5)$$

$$\int_0^{1-x_1} dx_2 \{D_{f_1 q}(x_1, x_2) - D_{f_1 \bar{q}}(x_1, x_2)\} = (N_q - \delta_{f_1 q} + \delta_{f_1 \bar{q}}) D_{f_1}(x_1). \quad (6)$$

Notice that if the DPDFs are parton exchange symmetric,

$$D_{f_1 f_2}(x_1, x_2) = D_{f_2 f_1}(x_2, x_1), \quad (7)$$

the sum rules with respect to the first parton imply the sum rules with respect to the second one since the evolution equations also conserve parton exchange symmetry.

We see that the above sum rules relate the double and single parton distribution functions, which reflects the common origin of those distributions, namely, the expansion of the nucleon state in Fock light-cone components [14]. In addition, the sum rules for the single parton distributions are also satisfied - the momentum sum rule

$$\sum_f \int_0^1 dx x D_f(x) = 1 \quad (8)$$

and the quark valence sum rule for $q = u, d, s$

$$\int_0^1 dx \{D_q(x) - D_{\bar{q}}(x)\} = N_q. \quad (9)$$

III. MELLIN MOMENT FORMULATION

Let us perform the double Mellin transform of the DPDFs

$$\tilde{D}_{f_1 f_2}(n_1, n_2) = \int_0^1 dx_1 \int_0^1 dx_2 (x_1)^{n_1-1} (x_2)^{n_2-1} D_{f_1 f_2}(x_1, x_2) \Theta(1 - x_1 - x_2). \quad (10)$$

where $n_{1,2}$ are complex numbers and we omit the scale Q_0 in the notation from now on. The step function $\Theta(1-x_1-x_2)$ is inserted into the definition of the Mellin transform since this is the region over which the double parton distribution is defined. Similarly, for the single parton distribution functions, we define the Mellin moments

$$\tilde{D}_f(n) = \int_0^1 dx x^{n-1} D_f(x), \quad (11)$$

where n is a complex number. The Mellin moments can be transformed back to the x -space using the inverse transformation for the single parton distribution,

$$D_f(x_1) = \int_C \frac{dn}{2\pi i} (x_1)^{-n} \tilde{D}_f(n), \quad (12)$$

and similarly for the double parton distribution function

$$D_{f_1 f_2}(x_1, x_2) = \int_{C_1} \frac{dn_1}{2\pi i} (x_1)^{-n_1} \int_{C_2} \frac{dn_2}{2\pi i} (x_2)^{-n_2} \tilde{D}_{f_1 f_2}(n_1, n_2), \quad (13)$$

where the integration contours C_1 and C_2 lie to the right of the rightmost singularity in the complex plane of n_1 and n_2 , respectively. Let us emphasize that formula (13) is only applicable to $x_{1,2} \in [0, 1]$ and $x_1 + x_2 \leq 1$.

The sum rules (3) and (4) can be written with the help of the Mellin moments after the integration of both sides over x_2 with the factor $(x_2)^{n_2-1}$. Thus, we find

$$\sum_{f_1} \tilde{D}_{f_1 f_2}(2, n_2) = \tilde{D}_{f_2}(n_2) - \tilde{D}_{f_2}(n_2 + 1), \quad (14)$$

$$\tilde{D}_{q f_2}(1, n_2) - \tilde{D}_{\bar{q} f_2}(1, n_2) = (N_q - \delta_{f_2 q} + \delta_{f_2 \bar{q}}) \tilde{D}_{f_2}(n_2). \quad (15)$$

Analogous relations hold true for the second parton

$$\sum_{f_2} \tilde{D}_{f_1 f_2}(n_1, 2) = \tilde{D}_{f_1}(n_1) - \tilde{D}_{f_1}(n_1 + 1), \quad (16)$$

$$\tilde{D}_{f_1 q}(n_1, 1) - \tilde{D}_{f_1 \bar{q}}(n_1, 1) = (N_q - \delta_{f_1 q} + \delta_{f_1 \bar{q}}) \tilde{D}_{f_1}(n_1). \quad (17)$$

These sum rules have to be satisfied simultaneously with the momentum sum rule for the single parton distribution

$$\sum_f \tilde{D}_f(2) = 1, \quad (18)$$

and the valence quark sum rule

$$\tilde{D}_q(1) - \tilde{D}_{\bar{q}}(1) = N_q. \quad (19)$$

It would be extremely useful to construct initial conditions for DPDFs which fulfill the above sum rules since the PDFs on the r.h.s of Eqs. (3)-(6) are very well known from the global analysis fits. Thus, the PDFs constrain the DPDFs, solving or significantly reducing the problem of uncertainty in the specification of initial conditions for DPDFs evolution. For this purpose, we consider the single PDF parametrization from the MSTW fits [39]. We will choose the leading order (LO) version of this parameterization since the evolution equations (2) are given in leading logarithmic approximation. We start from considering the case with only gluons. This limits the set of the possible distributions in which D_{gg} and D_g are the only relevant functions and of course we only have to fulfill the momentum sum rule.

IV. PURE GLUON CASE

The single gluon distribution is specified in the LO MSTW parameterization at the scale $Q_0 = 1$ GeV and is given in the form

$$D_g(x) = A_g x^{\delta_g - 1} (1 - x)^{\eta_g} (1 + \epsilon_g \sqrt{x} + \gamma_g x), \quad (20)$$

where $A_g = 0.0012216$, $\delta_g = -0.83657$, $\eta_g = 2.3882$, $\epsilon_g = -38.997$ and $\gamma_g = 1445.5$. For errors on these parameters and the discussion of their determination, see [39]. Since we only use gluons in our analysis, we renormalize the gluon distribution such that the total longitudinal momentum carried by gluons equals one, which results in $A_g = 0.0033723$. This is really not so crucial here as the normalization can be set arbitrarily for the single channel case and it does not affect the subsequent discussion. The parametrization (20) can be written in a general form which is more suitable for our purpose

$$D_g(x) = \sum_{k=1}^L N_g^k x^{\alpha_g^k} (1 - x)^{\beta_g^k}, \quad (21)$$

where $L = 3$ and the parameters N_g^k , α_g^k and β_g^k can easily be found by the comparison with eq. (20),

$$\begin{aligned} N_g^1 &= A_g, & N_g^2 &= \epsilon_g A_g, & N_g^3 &= \gamma_g A_g \\ \alpha_g^1 &= \delta_g - 1, & \alpha_g^2 &= \delta_g - \frac{1}{2}, & \alpha_g^3 &= \delta_g, & \beta_g^1 &= \beta_g^2 = \beta_g^3 = \eta_g. \end{aligned} \quad (22)$$

In the Mellin space, the gluon distribution (20) can be written as

$$\tilde{D}_g(n) = \sum_{k=1}^L N_g^k \frac{\Gamma(n + \alpha_g^k) \Gamma(\beta_g^k + 1)}{\Gamma(n + \alpha_g^k + \beta_g^k + 1)}. \quad (23)$$

where the expression on the r.h.s., $\Gamma(x)\Gamma(y)/\Gamma(x+y) \equiv B(x,y)$, is the Euler Beta function. Thus the MSTW parametrization for the initial condition is in the form of the sum over the Beta functions with different sets of parameters which govern the small $x \rightarrow 0$ and large $x \rightarrow 1$ behavior.

For the double parton distribution D_{gg} we shall take the following ansatz in the form

$$D_{gg}(x_1, x_2) = \sum_{k=1}^L \bar{N}_{gg}^k (x_1 x_2)^{\bar{\alpha}_g^k} (1 - x_1 - x_2)^{\bar{\beta}_g^k}, \quad (24)$$

where \bar{N}_{gg}^k , $\bar{\alpha}_g^k$ and $\bar{\beta}_g^k$ are the parameters to be determined. The above ansatz is in the form of the sum over the Dirichlet-type distributions of order $K = 3$

$$f(x_1, x_2; \gamma_1, \gamma_2, \gamma_3) = N \prod_{i=1}^{K=3} x_i^{\gamma_i}, \quad (25)$$

with $x_1, x_2 > 0$, $x_1 + x_2 \leq 1$ and $x_3 = 1 - x_1 - x_2$. Notice that the function (24) is parton exchange symmetric, $D_{gg}(x_1, x_2) = D_{gg}(x_2, x_1)$, as it should be. It should also fulfill the momentum sum rules with respect to both partons. Also note that the number of terms in this sum, L , is the same as the number of terms in the single parton distribution (21). The Mellin space representation of the above ansatz reads

$$\tilde{D}_{gg}(n_1, n_2) = \sum_{k=1}^L \bar{N}_{gg}^k \frac{\Gamma(n_1 + \bar{\alpha}_g^k) \Gamma(n_2 + \bar{\alpha}_g^k) \Gamma(1 + \bar{\beta}_g^k)}{\Gamma(n_1 + n_2 + 1 + \bar{\beta}_g^k + 2\bar{\alpha}_g^k)}, \quad (26)$$

which is in the form of the generalized Beta function.

In the pure gluon case only the momentum sum rule for the DPDFs in the momentum space reads,

$$\int_0^{1-x_2} dx_1 x_1 D_{gg}(x_1, x_2) = (1 - x_2) D_g(x_2), \quad (27)$$

and similarly for the momentum sum rule with respect to the second gluon. In the Mellin representation this condition reduces to

$$\tilde{D}_{gg}(2, n_2) = \tilde{D}_g(n_2) - \tilde{D}_g(n_2 + 1), \quad (28)$$

It is easy to see that the distributions of the form presented in Eqs. (21) and (24) fulfill the momentum sum rule provided certain constraints are satisfied. The right hand side of Eq. (28) is the difference of the moments of single parton distributions which can be written as

$$\tilde{D}_g(n_2) - \tilde{D}_g(n_2 + 1) = \sum_{k=1}^L N_g^k B(n_2 + \alpha_g^k, \beta_g^k + 2) = \sum_{k=1}^L N_g^k \frac{\Gamma(n_2 + \alpha_g^k) \Gamma(\beta_g^k + 2)}{\Gamma(n_2 + \alpha_g^k + \beta_g^k + 2)}, \quad (29)$$

where we used the following property of the Beta function

$$B(a, b) = B(a + 1, b) + B(a, b + 1). \quad (30)$$

On the other hand the left-hand-side of Eq. (28) is obtained upon setting $n_1 = 2$ in Eq. (26)

$$\tilde{D}_{gg}(2, n_2) = \sum_{k=1}^L \bar{N}_{gg}^k \frac{\Gamma(2 + \bar{\alpha}_g^k) \Gamma(n_2 + \bar{\alpha}_g^k) \Gamma(1 + \bar{\beta}_g^k)}{\Gamma(3 + n_2 + \bar{\beta}_g^k + 2\bar{\alpha}_g^k)}. \quad (31)$$

Now in order for the momentum sum rule to be satisfied, we need Eqs. (29) and (31) to be equal term by term in the sum over k . From the requirement that the poles and zeros in n in each term should be in the same location we find that

$$\bar{\alpha}_g^k = \alpha_g^k, \quad 2\bar{\alpha}_g^k + \bar{\beta}_g^k + 3 = \alpha_g^k + \beta_g^k + 2, \quad (32)$$

and from the requirement that the normalization of each terms should be the same we have that

$$\bar{N}_{gg}^k \Gamma(2 + \bar{\alpha}_g^k) \Gamma(1 + \bar{\beta}_g^k) = N_g^k \Gamma(2 + \beta_g^k). \quad (33)$$

From these relations we compute all the parameters of the double gluon distribution in terms of the known parameters of the single gluon distribution, given by Eq. (22), to find the following parameter-free double distribution at the initial scale $Q_0 = 1$ GeV,

$$D_{gg}(x_1, x_2) = \sum_{k=1}^3 N_g^k \frac{\Gamma(\beta_g^k + 2)}{\Gamma(\alpha_g^k + 2) \Gamma(\beta_g^k - \alpha_g^k)} (x_1 x_2)^{\alpha_g^k} (1 - x_1 - x_2)^{\beta_g^k - \alpha_g^k - 1}, \quad (34)$$

satisfying the momentum sum rule (27) by construction. Notice that even for small momentum fractions, $x_{1,2} \ll 1$, the resulting double gluon distribution is not factorizable, i.e. $D_{gg}(x_1, x_2) \neq D_g(x_1) D_g(x_2)$.

V. EVOLUTION OF DOUBLE GLUON DISTRIBUTION

The evolution equations (2) reduced to the pure gluon case have the following form

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} D_{gg}(x_1, x_2, Q) = \frac{\alpha_s(Q)}{2\pi} \left\{ \int_{x_1}^{1-x_2} \frac{du}{u} \mathcal{P}_{gg}\left(\frac{x_1}{u}\right) D_{gg}(u, x_2, Q) \right. \\ + \int_{x_2}^{1-x_1} \frac{du}{u} \mathcal{P}_{gg}\left(\frac{x_2}{u}\right) D_{gg}(x_1, u, Q) \\ \left. + \frac{1}{x_1 + x_2} P_{gg}^R\left(\frac{x_1}{x_1 + x_2}\right) D_g(x_1 + x_2, Q) \right\}. \end{aligned} \quad (35)$$

where P_{gg}^R is the gluon-to-two gluon splitting function for real emission in the LO approximation. Strictly speaking, such an equation can be reasonable approximation for small values of momentum fractions. Using our numerical program, we solve the above equation with the initial condition (34). We compare our results with those obtained from the usually assumed form of the initial conditions [18], which satisfy the momentum sum rule only approximately,

$$D_{gg}(x_1, x_2) = D_g(x_1) D_g(x_2) \rho(x_1, x_2) \quad (36)$$

where the correlation factor

$$\rho(x_1, x_2) = \frac{(1 - x_1 - x_2)^2}{(1 - x_1)^2 (1 - x_2)^2}. \quad (37)$$

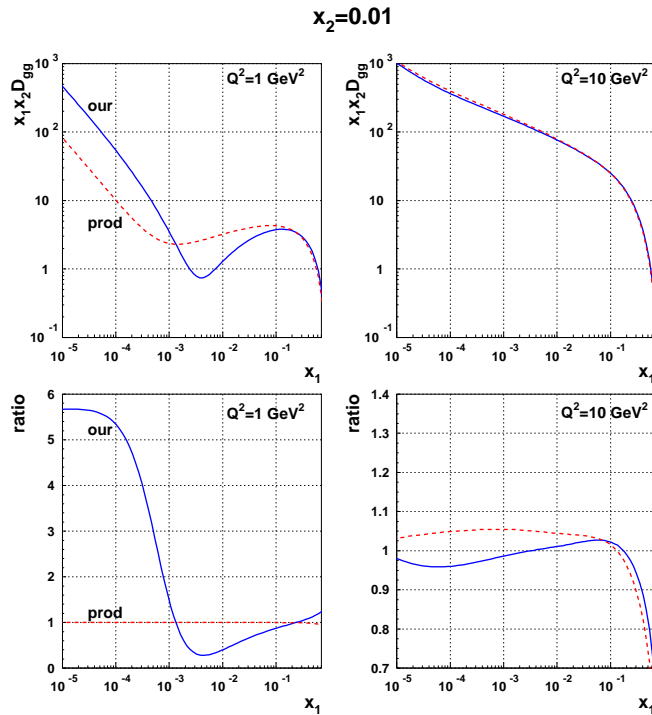


FIG. 1: The distribution $x_1 x_2 D_{gg}(x_1, x_2 = 10^{-2})$ at $Q_0^2 = 1 \text{ GeV}^2$ (left upper panel) and $Q^2 = 10 \text{ GeV}^2$ (right upper panel) and the ratio (38) (lower panels). The solid lines correspond to input (34) (our) while the dashed lines to input (36) (prod).

The results are shown in Fig. 1 and Fig. 2. We plot there the double gluon distribution $x_1 x_2 D_{gg}(x_1, x_2)$ as a function of x_1 for two values of the scale, initial $Q_0^2 = 1 \text{ GeV}^2$ and $Q^2 = 10 \text{ GeV}^2$ (upper panels), for two fixed values of x_2 , small 10^{-2} and large 0.5 , respectively. The solid lines show the results obtained from our input (34) while the dashed lines correspond to the input (36) with the gluon distribution (20). In the lower panels we plot the ratio,

$$\text{ratio} = \frac{D_{gg}(x_1, x_2)}{D_g(x_1)D_g(x_2)}, \quad (38)$$

which characterizes factorizability of the double gluon distribution into a product of two single gluon distributions.

For both values of x_2 , the initial double gluon distributions differ significantly for small values of x_1 , up to 10^{-1} for $x_2 = 10^{-2}$ and up to 10^{-3} for $x_2 = 0.5$. However, the QCD evolution equation erases this difference already at the scale $Q^2 = 10 \text{ GeV}^2$, see the upper panels in both figures. As we have observed, the initial distribution (34) is not factorizable into a product of two single gluon distributions for any values of x_1 and x_2 . However, if both momentum fractions are small (< 0.01), D_{gg} becomes factorizable with good accuracy after evolution to the shown value of Q^2 , see the lower panels in both figures. A small breaking of the factorization can be attributed to the non-homogeneous term in the evolution equation (35). If one of the two momentum fractions is large, like the shown $x_2 = 0.5$, this is no longer the case and the factorization is significantly broken for all values of x_1 independent of the values of the evolution scale. We check that for larger values of Q^2 than shown here. We have to remember, however, that the large x domain has to be supplemented by quarks.

In conclusion, the initial double gluon distribution (34) is very different from that proposed so far. However, the QCD evolution equation significantly diminishes this difference at not so high values of the evolution scale Q^2 .

VI. SUMMARY

In this paper we constructed the double gluon distribution D_{gg} from the known single gluon distribution D_g , given by the MSTW parameterization, in the pure gluon case. The construction is based on the expansion in terms of functions which are the Dirichlet distributions. Since the MSTW distribution has already the form of the sum over Dirichlet distributions of order 2, we postulated the double gluon distribution as a sum over the Dirichlet distributions

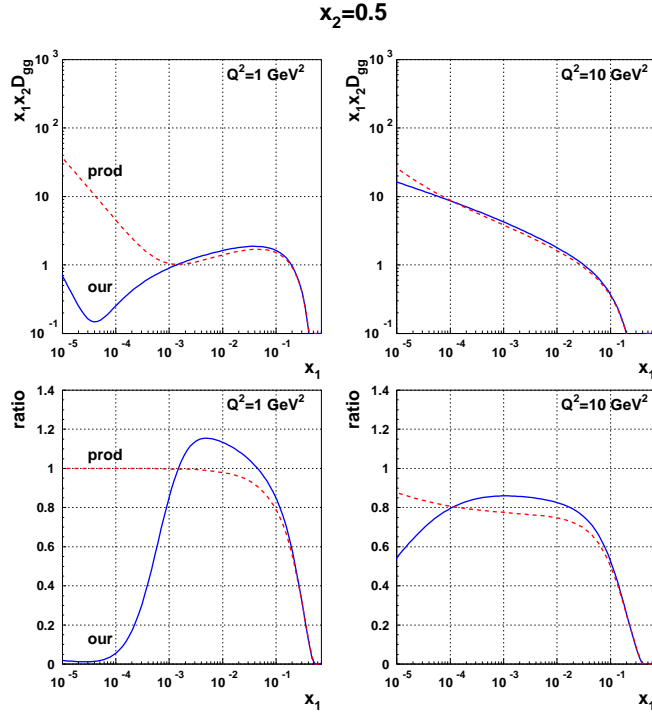


FIG. 2: The distribution $x_1 x_2 D_{gg}(x_1, x_2 = 0.5)$ at $Q_0^2 = 1 \text{ GeV}^2$ (left upper panel) and $Q^2 = 10 \text{ GeV}^2$ (right upper panel) and the ratio (38) (lower panels). The solid lines correspond to input (34) (our) while the dashed lines to input (36) (prod).

of order 3 and identified the parameters in the two distributions using the momentum sum rule for this purpose. As a result, we obtained the parameter free double gluon distribution which we evolve with the QCD evolution equation. We studied the build up of the factorized form of D_{gg} with the increasing evolution scale Q . We found that such a form approximately sets up rather quickly for small momentum fractions, $x_{1,2} < 0.1$. As expected, for higher values of x , the factorized form is not valid at all due to the constraint from the momentum sum rule.

The next step would be to extend this formalism to include the quarks and satisfy the momentum and valence quark sum rules simultaneously. The expansion in terms of the Dirichlet functions can be constructed also in the case with quarks. The whole formalism is however much more complicated due to the large number of the double parton distribution and therefore the full analysis will be presented in the future publication.

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- [1] Axial Field Spectrometer Collaboration, T. Akesson *et al.*, Z.Phys. **C34**, 163 (1987).
 - [2] CDF Collaboration, F. Abe *et al.*, Phys.Rev.Lett. **79**, 584 (1997).
 - [3] CDF Collaboration, F. Abe *et al.*, Phys.Rev. **D56**, 3811 (1997).
 - [4] D0 Collaboration, V. Abazov *et al.*, Phys.Rev. **D81**, 052012 (2010), [0912.5104].
 - [5] ATLAS Collaboration, G. Aad *et al.*, New J.Phys. **15**, 033038 (2013), [1301.6872].
 - [6] CMS Collaboration, S. Chatrchyan *et al.*, JHEP **1403**, 032 (2014), [1312.5729].

- [7] ATLAS Collaboration, G. Aad *et al.*, JHEP **1404**, 172 (2014), [1401.2831].
- [8] V. Shelest, A. Snigirev and G. Zinovev, Phys.Lett. **B113**, 325 (1982).
- [9] G. Zinovev, A. Snigirev and V. Shelest, Theor.Math.Phys. **51**, 523 (1982).
- [10] R. K. Ellis, W. Furmanski and R. Petronzio, Nucl.Phys. **B212**, 29 (1983).
- [11] A. Bukhvostov, G. Frolov, L. Lipatov and E. Kuraev, Nucl.Phys. **B258**, 601 (1985).
- [12] A. M. Snigirev, Phys. Rev. **D68**, 114012 (2003), [hep-ph/0304172].
- [13] V. L. Korotkikh and A. M. Snigirev, Phys. Lett. **B594**, 171 (2004), [hep-ph/0404155].
- [14] J. R. Gaunt and W. J. Stirling, JHEP **03**, 005 (2010), [0910.4347].
- [15] B. Blok, Y. Dokshitzer, L. Frankfurt and M. Strikman, Phys.Rev. **D83**, 071501 (2011), [1009.2714].
- [16] F. A. Ceccopieri, Phys. Lett. **B697**, 482 (2011), [1011.6586].
- [17] M. Diehl and A. Schafer, Phys. Lett. **B698**, 389 (2011), [1102.3081].
- [18] J. R. Gaunt and W. J. Stirling, JHEP **1106**, 048 (2011), [1103.1888].
- [19] M. Ryskin and A. Snigirev, Phys.Rev. **D83**, 114047 (2011), [1103.3495].
- [20] J. Bartels and M. G. Ryskin, 1105.1638.
- [21] B. Blok, Y. Dokshitzer, L. Frankfurt and M. Strikman, Eur.Phys.J. **C72**, 1963 (2012), [1106.5533].
- [22] M. Diehl, D. Ostermeier and A. Schafer, JHEP **1203**, 089 (2012), [1111.0910].
- [23] M. Luszczak, R. Maciula and A. Szczurek, Phys. Rev. **D85**, 094034 (2012), [1111.3255].
- [24] A. V. Manohar and W. J. Waalewijn, Phys.Rev. **D85**, 114009 (2012), [1202.3794].
- [25] M. Ryskin and A. Snigirev, Phys.Rev. **D86**, 014018 (2012), [1203.2330].
- [26] J. R. Gaunt, JHEP **1301**, 042 (2013), [1207.0480].
- [27] B. Blok, Y. Dokshitzer, L. Frankfurt and M. Strikman, Eur.Phys.J. **C74**, 2926 (2014), [1306.3763].
- [28] A. van Hameren, R. Maciula and A. Szczurek, Phys. Rev. **D89**, 094019 (2014), [1402.6972].
- [29] R. Maciula and A. Szczurek, Phys. Rev. **D90**, 014022 (2014), [1403.2595].
- [30] A. Snigirev, N. Snigireva and G. Zinovev, Phys.Rev. **D90**, 014015 (2014), [1403.6947].
- [31] K. Golec-Biernat and E. Lewandowska, Phys.Rev. **D90**, 094032 (2014), [1407.4038].
- [32] J. R. Gaunt, R. Maciula and A. Szczurek, Phys. Rev. **D90**, 054017 (2014), [1407.5821].
- [33] L. A. Harland-Lang, V. A. Khoze and M. G. Ryskin, J. Phys. **G42**, 055001 (2015), [1409.4785].
- [34] B. Blok and M. Strikman, Eur. Phys. J. **C74**, 3214 (2014), [1410.5064].
- [35] R. Maciula and A. Szczurek, 1503.08022.
- [36] R. Kirschner, Phys.Lett. **B84**, 266 (1979).
- [37] K. Golec-Biernat and E. Lewandowska, Phys.Rev. **D90**, 014032 (2014), [1402.4079].
- [38] W. Broniowski and E. Ruiz Arriola, Few Body Syst. **55**, 381 (2014), [1310.8419].
- [39] A. Martin, W. Stirling, R. Thorne and G. Watt, Eur.Phys.J. **C63**, 189 (2009), [0901.0002].
- [40] A. V. Manohar and W. J. Waalewijn, Phys.Lett. **B713**, 196 (2012), [1202.5034].
- [41] B. Blok, M. Strikman and U. A. Wiedemann, Eur. Phys. J. **C73**, 2433 (2013), [1210.1477].